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Measuring the Earth

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Finding the size of our planet does not require sophisticated technology, or distant travel as in ancient times. You can accomplish this task using meter sticks, a stopwatch, and some math. The process is described below, followed by numeric examples, error analysis, enrichment concepts, and notes for teachers.

In this activity students watch a bright star (but not a planet!) disappear behind a distant horizon. They start a timer when the star disappears as seen from very close to the ground and stop the timer when the star disappears as seen from a slightly higher location. The elapsed time is used to calculate the Earth's diameter.

Choose a setting: Do these measurements in a stable place that gives you an unobstructed view of a stable and distant horizon to the west.

If your location is not stable (such as being in a small boat subject to wave action) or the horizon point you choose is not stable (such as the top of a tree disturbed by wind) each measurement could be erroneous, producing unacceptable errors in the computed result.

Choose a horizon point as close to due west as possible, so that your direction of view will be approximately opposite to the direction of the earth's rotation.

Choose a horizon point as far away from you as possible so that changing your body's position will not change the vertical angle from your eyes to the horizon. As an example; if a person 5' 3" tall uses a horizon point a mile away, that height is only 0.1 percent of that distance, equivalent to half the thickness of this line compared to its length:

Figure 1. Accurate scale drawing of an angle of 1/1000 radian. This angle is small enough to permit the analytical approximations used below.

That tiny angle will allow us to use the analytical approximations below. Furthermore, that percentage is much less than the percentage errors that could be induced by reasonable mistakes in our measurements.

But a very short distance to the horizon will create a much bigger change of vertical angle. For example, consider what happens if a person 6' tall uses the roof line of a three-story building a block away. Then we have an object about 35' high approximately 500' away. The building and the person look like this:

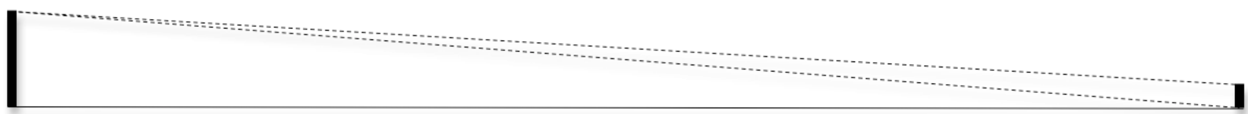

Figure 2. Accurate scale drawing of a person 6 feet tall and a building 35 feet high, separated by a distance of 500 feet. The difference between the two viewing angles represented by the dotted lines is too large to conform to the analytical approximations used in this analysis.

The two dashed lines represent the person's line of sight lying down and standing up, and the angle between those two lines is too big to permit the analytical approximations used below. That angle also creates a measurement error. The person's height is 6/500 = 1.2 percent of the distance to the building. This is more than double the percentage of error that might be induced by a measurement we might make, and therefore unacceptable for our purposes.

If you choose a horizon point that is very far away, your line of sight to that point does not have to be level; it can be any angle above a level line. Consider two people working from the same place and using the crest line of a far-distant mountain range for their horizon points. Suppose one person uses a star that will set behind the highest peak and the other person uses a star that will set in a very low mountain pass. Then the person using the peak has the vertical line of sight represented by the dashed line in Figure 3 and the person using the pass has the vertical line of sight represented by the tilted solid line in the same figure.

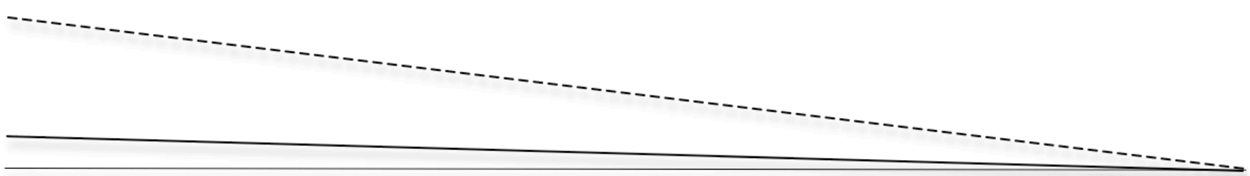

Figure 3. A perfectly level viewing line, a slightly higher viewing angle (the higher solid line) and a significantly higher viewing angle (the dashed line). All three viewing angles rotate with the earth at the same speed. If all three observers use equally distant horizon points, all three viewing angles are equally useful for this activity.

There are some subtleties here, but each person uses starting and final lines of sight that are close enough together to be acceptable for all our purposes. Each person's horizon point rotates with the earth through the same arc in the same amount of time. Therefore, each horizon point is as acceptable as a level horizon point at the same distance would be.

To summarize: The distance to the horizon point is very important and should be maximized, but the vertical angle to the horizon point does not matter at all.

Measure time:

Lie on a firm flat surface and watch a star disappear below the horizon. At the moment when the star disappears, start a timer. Quickly stand up. Again watch the star disappear. At the moment when the star disappears, stop the timer.

If standing up is too difficult, too uncomfortable, or too slow, you can begin by kneeling or sitting in a chair instead of lying down.

This measurement must use a star. The last limb of the setting sun must NOT be used because of the risk of eye damage. The last limb of the setting moon cannot be used because of the moon's motion relative to the earth; students should prove this as an exercise.

Measure height:

Have a partner measure the height from the surface to the center of the pupil of either of your eyes, just before you start the timer and just after you stop it. The difference between those two numbers is your effective height difference, labeled h in the drawing below.

Analyze the geometry:

In Figure 4 below, the radius of the Earth is r. During the time you measured, the Earth's radius beneath you moved from its starting location r_s to its final location r_f, sweeping out angle θ. Your line of site moved from L_s to L_f, also sweeping out angle θ.

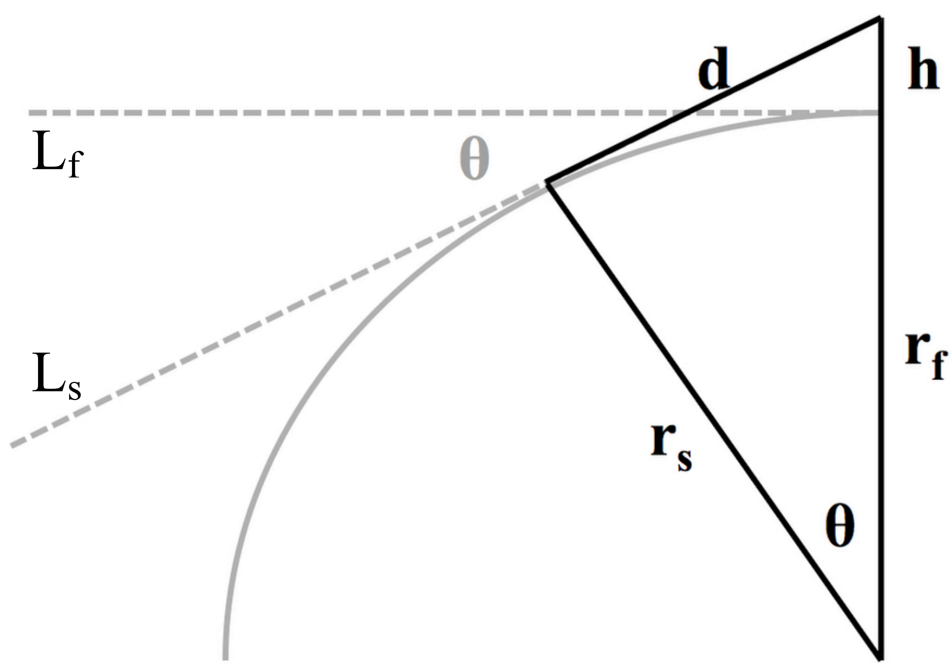

Figure 4. In gray: curvature of the Earth's surface, starting line of sight, and final line of sight. The Earth's rotation and angle θ it causes are exaggerated for clarity. In black: geometric construction to enable the analysis.

Construct the right triangle dr_sr_f. The Pythagorean Theorem tells us $d^2 + r_s^2 = (r_f + h)^2$ so $d^2 + r_s^2 = r_f^2 + 2r_fh + h^2$. We know $r_s = r_f$ so we can simplify to $d^2 + r^2 = r^2 + 2rh + h^2$. Since $h << r$, h^2 is negligible in comparison to r^2 and so can be ignored. Then we have $d^2 = 2rh$.

$d/r = \tan\theta$ so $d = r\tan\theta$. When θ is measured in radians (360 degrees = 2π radians) and is tiny, $\tan\theta = \theta$ so now we have $d = r\theta$ and therefore $d^2 = r^2\theta^2$. Immediately above we proved $d^2 = 2rh$ so now we know
$r^2\theta^2 = 2rh$. Then our final result is $r = 2h/\theta^2$. That is our formula for the Earth's radius.

Analyze the proportions:

Write the proportionality between the amount the Earth rotated and the time that took:

$$\frac{\theta \text{ degrees}}{360 \text{ degrees}} = \frac{t \text{ seconds}}{24 \text{ hours}} = \frac{t \text{ seconds}}{24 \times 60 \times 60 \text{ seconds}} \quad \text{so} \quad \theta = \frac{360 \text{ deg } t \text{ sec}}{24 \times 60 \times 60} = \frac{t \text{ sec}}{240}$$

When t is measured in seconds and θ is measured in degrees, $\theta = t/240$. So when θ is measured in radians that becomes $\theta = (t/240) \times (2\pi/360)$ so $\theta^2 = 5.2885 \times 10^{-9}\, t^2$. Then the Earth's diameter is given by $D = 2r = 4h/\theta^2 = 756{,}358{,}583.086\, h/t^2$ where t is measured in seconds.

Numeric examples:

Suppose a person's effective height difference is 5'6" (measured as 1.68 meters) and the time is measured as 10 seconds. Then the Earth's calculated diameter is
$D = 756{,}358{,}583.1\, h/t^2 = 756{,}358{,}583.1 \times 1.68 / 10^2 = 12{,}706{,}824.2$ meters. The Earth is not a perfect sphere; the accepted value for its average diameter is 12,742,000 meters. So the above measurements produce a calculated error of 0.28 percent.

If a person's effective height difference is 6' (measured as 1.83 meters) and the time is 10.5 seconds, the Earth's diameter is calculated to be 12,554,523.4 meters, which is an error of 1.47 percent.

Error analysis:

Measuring the two heights might include small errors. The error in measuring lying height and the error in measuring standing height add up to the error in h. If both errors are in the same direction and of the same size, there will be no error in h. If the errors are in different directions and/or of different sizes, the errors will add up to an error in h. Suppose a person's true height difference is h_1 but is recorded as $h_2 \neq h_1$. Since the Earth's diameter D is a linear function of the effective height difference h, the percent error in h will produce the same percent error in D.

An error of ½ percent in h will create an error of ½ percent in D. For a person 1.8 meters tall, that would be a total of 9 millimeters of non-offsetting errors in measuring lying height and standing height. That accuracy should be easy to achieve.

Starting and stopping the timer might create small errors. A person's motor reaction time to similar visual stimuli (when the stimuli are expected, the stimuli are simple, and the desired response is simple) is an almost-constant delay. Therefore the recorded time should be very close to the actual time. Still, there might be a small error in the person's behavior or an inaccuracy in the timer (such as rounding to the nearest tenth of a second instead of giving an exact reading). Suppose there is an error and the true time t is recorded as t+p where the error p is either positive or negative but not zero. D is an inverse quadratic function of t, so the percent error in D will be as follows. Students should prove this as an exercise.

$$\text{percent error in D} = -\frac{2tp + p^2}{t^2 + 2tp + p^2}$$

The effect of the error depends on the correct time as well as on the error. To get some perspective on this effect, let's rework an above example by injecting a time error of 0.2 seconds.

Suppose h = 5'6" as before but t = 10.2 seconds instead of 10.
Then D = 12,190,351.9 meters and the error is 4.33% instead of 0.28% (a shift of 4.05%).

Suppose h = 5'6" as before but t = 9.8 seconds instead of 10.
Then D = 13,205,791.4 meters and the error is 3.64% instead of 0.28% (a shift of 3.36%).

Thus we see that the impact of the timing error depends on the error's direction as well as its size. These inconvenient characteristics are caused by the inverse quadratic relationship.

Modify the experiment:

We saw above that a reasonably small error in t has a greater impact than a reasonably small error in h. Can you think of a way to modify this experiment such that any error in t is a smaller percentage of t ? Discuss this with your lab partner, then carry out your modified experiment and compute your percentage of error in the diameter of our planet. Did your accuracy improve?

If none of the lab teams in your class can think of a modification, consider this hint. Suppose someone has a medical condition that confines that person to a wheelchair. Further suppose that the timer is started with the person seated, then two people help the person stand up to complete the experiment. If the person's eyes are 4'1" up when seated and 5'6" up when standing, then h = 0.4318 meters. Then if recorded time is 5.05 seconds, D = 12,806,442.09 meters and the error is 0.51 percent. But if the time is erroneously recorded as 4.85 seconds the error becomes 8.97 percent. Compare those results to the preceding examples.

Develop a general principle:

After you have modified the experiment and achieved more accurate results, explain why your modification improved accuracy. Now think about doing an original experiment where the true result is not yet known. What general principle(s) related to measurement should you use in order to increase your confidence in the accuracy of your results?

Notes to teachers - not using the moon:

The moon revolves 360 degrees about the earth in 28 days. The earth rotates on its axis 360 degrees in 1 day. Therefore, the moon does not appear to move through the night sky at the same speed as the stars. This error of 1/28 is about 3½ percent. That is triple the error induced by reasonable errors in measuring the height difference, and as large as an error induced by timing errors of 0.2 seconds. Using the moon is guaranteed to at least double - and perhaps quadruple - overall error. Therefore the moon cannot be used for these measurements.

Notes to teachers - precision and errors:

Increasing the precision of measurement methods and instruments decreases absolute error.

Enlarging what you measure decreases that absolute error as a percentage of the measured value.

Practicing reduces human error and variability, and so decreases absolute and relative error.

Measuring many times will either:
1. get the same result and thus increase confidence in your accuracy, or
2. show small variations which will let you "average out" your reported measurement, or
3. show large variations which indicate you need to improve your methods.

Tell students to repeat their measurements for several stars during the same work session and then compute the planet's diameter for each star used. Have them evaluate their precision and consistency. This will identify outlying results, which indicate measurement errors and so can be eliminated from the experiment's findings. The remaining results should be very similar, allowing a determination of the final diameter to be reported as "the" result of the person's efforts. Then compare all of the students' results and look for outliers among them.
When students use their pocket calculators the results will be affected by truncation errors because of the devices' size limits. So their numbers might differ slightly from the numbers shown, which were computed by Microsoft Excel to many decimal places and rounded here for convenience.

Notes to teachers - atmospheric problems:

These measurements should be postponed until better "seeing" is available if any of the following conditions exist:

1. Cloud cover near the horizon point.

2. Any form of precipitation.
3. A dust storm or dust devil near the horizon point.
4. Enough haze to "fog" or "cloud" images at the distance of the horizon.

Notes to teachers - reaction times:

Reaction times are involved in starting and stopping the timer. Each event:
1. Is triggered by the same simple event which is understood and anticipated as to location and approximate time.
2. Is a simple response to the perceived stimulus.
3. Employs the same person's identical perceptual, cognitive, and motor response mechanisms in the same way.
4. Uses the same tool in the same way, with the same amount of accuracy/error in the tool.

Therefore any person who is not impaired as to perception, cognition, motor response, or understanding of the task at hand should demonstrate a uniform and very small delay between the moment of the stimulus and the moment the response is completed. Since the same delay exists when starting the timer and stopping the timer, the measured time should be very accurate.

Those assertions have been proven by experiments which measured the size and consistency of reaction times for many hundreds of university students in beginning physics laboratories. Each semester, our students' first lab experiment was one designed to measure their reaction times. The stimulus was a simple, understood, and anticipated optical signal. The response was to press the stem of a stopwatch upon the first stimulus and again upon the second stimulus. For every student, the measured time lapse was consistently close to the known predetermined time between stimuli. Errors of 0.0 or 0.1 seconds were common. Errors greater than 0.2 seconds were rare.

Having students practice with timers in the classroom will improve their skills, their confidence, and the consistency of the times they measure and report.