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Why Tailgating on Freeways is Unsafe: A Real-life Example Using Quadratic Equations

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"When are we ever going to use this?" is a question teachers usually face when teaching math, especially when teaching topics such as the quadratic equation. Science teachers, too, face such questions, but generally not to the same extent as math teachers. Here, we will discuss one use of the quadratic equation, and also indicate some other factors (including science related factors) that also need to be taken into account to give a more accurate/appropriate solution. Specifically, we discuss how to use a quadratic equation to compute the stopping distance of a car traveling at a given speed, and then discuss other ways to compute the stopping distance.

First, a review of some equations of motion is in order. Although it is true that that these equations are strictly true for particles under constant acceleration and where motion is constrained to a straight line, it is common to approximate real life situations with mathematical models based on ideal situations and come to reasonably appropriate solutions.

For initial velocity *u*, and constant acceleration *a*, the final velocity *v*, after time *t* is given by

$$v = u + at \qquad \text{[equation 1]}$$

and the distance traveled, *s*, under these conditions, is given by

$$s = ut + \frac{1}{2}at^2 \qquad \text{[equation 2]}$$

Equation 2 is a *quadratic equation* connecting *u*, *a* and *t*. This equation can now be solved to find out the stopping distance of a car traveling at a given speed *u*, after time *t*, under a constant deceleration of – *a,* by applying the brakes.

If the car were traveling at initial speed *u*, and comes to a stop (when *v* = 0) under a constant deceleration –*a*, we can arrive at a 3rd equation (also quadratic), by solving for *t*, that gives the stopping distance, *s*:

$$s = \frac{u^2}{2a} \qquad \text{[equation 3]}$$

For example, say a 2011 BMW 335*is* can come to a stop in 4.1 seconds from a speed of 60 mph (= 88 fps), with a constant deceleration of 21.5 fpsps (feet per second per second). Substituting the relevant values into equation 2 gives

$$s = 88 \times 4.1 - \tfrac{1}{2}(21.5)(4.1^2) = 360.8 - 180.7 = 180.1 \text{ ft}$$

If we substitute the relevant values into equation 3, we also get

$$s = \frac{88^2}{2 \times 21.5} = 180.1 \text{ ft}$$

In other words, by using either of the two quadratic equations (equations 2 or 3) we can answer the question, "How far will a car traveling at speed *u* go before coming to a stop, once the car brakes are applied?" We can also find how long it takes to travel a given distance (e. g., as the car skids to a stop) if the car were traveling at a speed of *u*, but we will not discuss that in this paper. However, for those interested in using different given stopping distances to solve quadratic equations, please refer to Appendix, Note 3 .

Now let us use some science, specifically physics. If we consider kinetic energy, KE, and work done, W, by the brakes in getting the car to come to a stop from a speed of *u*, we get

W = $F \times s$, where F = force used to come to a stop, and s = stopping distance, and
KE= $\frac{1}{2} m u^2$, where m = mass of car, and u = initial speed of car

Since the car comes to a stop, *the work done, W, in stopping the car must equal the KE*. That is,

$$F \times s = \frac{1}{2} m u^2 \qquad \text{[equation 4]}$$

Now, equation 3 can be re-written as $s = c\,u^2$, where c is a constant, and equation 4 can also be re-written as $s = k\,u^2$, where k = a constant.

In either case, we can see that s, the *stopping distance, is proportional to* u^2, the initial speed, squared.

Let's look at the question below to see what this means.

If a car travels at 30 mph, and the stopping distance is 45 ft, under similar conditions, what would the stopping distance be, if the same car were traveling at 60 mph?

a) 45 ft b) 75 ft c) 90 ft d) 180 ft

Do you know that the correct answer is 180 ft? That is, when the speed doubles from 30 mph to 60 mph, the stopping distance quadruples, from 45 ft to 4 x 45 ft = 180 ft, or when the speed doubles (x 2), the stopping distance is 2^2 , or 4 times! Therefore, when the speed trebles from 30 mph to 90 mph, the stopping distance should be 3^2 x 45, or 9 x 45 = 405 ft.

This result should be sobering for motorists, as it predicts that *doubling* the speed *quadruples*, not doubles, the *stopping distance*. This quadratic equation demonstrates why a small increase in speed leads to a much larger increase in stopping distance, and why it makes sense to be more careful when driving in areas that might have more hazards, such as children crossing roads near schools, and why tailgating cars on the freeways is unsafe.

While we have worked out the stopping distance using the two quadratic equations above, it must be noted that, for most practical purposes, not all the relevant variables such as the constant deceleration *a*, or the time *t* taken to come to a stop, are readily available. As such, a useful approximation for the stopping distance is given by the following quadratic equation:

$s = \frac{u^2}{20}$, where *s* = stopping distance in feet, and *u* is the speed in mph.

(Please refer to the **Appendix, Note 1**, for further discussion of this approximation.)
Using the previous example of the 2011 BMW, we now have:

$s = \frac{60^2}{20}$ = 180 ft, which agrees pretty closely with our earlier results.

Remember that the stopping distance so far has been computed without taking into consideration other factors, such as the time taken to see the hazard (perception time), time taken to react to the hazard (reaction time), the condition of the road (whether wet, or asphalt surface, etc), the braking efficiency of the car brakes, the coefficient of friction, whether the car had passengers (thereby increasing the weight of the car), etc.

Even if we were to take just the *perception time* and *reaction time* into consideration, we should realize that the stopping distance would increase dramatically. For example, if a car were traveling at 60 mph (=88fps), and it takes ½ a sec to see a hazard (like seeing a person running across the road in front of the car), the car would have already traveled 44 ft. Then, assuming the reaction time is about 1 second (that is it takes a second to see the hazard, and then move the foot from the accelerator pedal to the brake pedal), the car would have traveled another 88 ft (even before the brake is actually applied).

Together then, the distance traveled (called the *thinking distance*) would be 132 ft, before the actual *stopping distance* due to the brakes, which in the case of the 2011 BMW we discussed earlier would be 180 ft. Adding the *thinking distance* to the *stopping distance*, we get 312 ft. In other words, if the BMW were traveling at 60 mph, it would travel about 312 ft before coming to a stop.

And, of course, if the perception time or reaction time is increased, due to a person's age, driving after consuming alcohol or drugs, texting or talking on a cell phone, etc., then the car would travel a much longer distance before coming to a stop, and avoiding the hazard. So, while car manufacturers might claim certain stopping distances—usually much smaller than those stated by other bodies interested in safe driving—it would be better to err on the side of overestimating than underestimating the total distance a car travels before it comes to a stop (please refer to **Appendix, Note 5**, for further discussion of published stopping distances, and factors affecting stopping distance).

For those interested in a slightly more accurate approximation of the total distance traveled, *d*, before a car traveling at speed *u*, comes to a complete stop (taking the reaction time as approximately 2/3 sec) the following formula can be used:

$d = u + \frac{u^2}{20}$ where *d* = distance in feet, and *u* = speed in mph.

(Please refer to **Appendix, Note 1**, for further discussion of this approximation.)
In the case of the BMW traveling at 60 mph, the stopping distance is given by

$d = 60 + \frac{60^2}{20} = 60 + 180 = 240$ ft

For an even more accurate approximation, the following formula can be used:

$d = 2.2\,u + \frac{u^2}{20}$ where *d* = distance in feet, and *u* = speed in mph.

(Please refer to **Appendix, Note 2**, for further discussion of this approximation.)
In the case of the BMW traveling at 60 mph, the total distance is given by

$$d = 2.2\, u + \frac{u^2}{20}$$

$$= 2.2\,(60) + \frac{60^2}{20}$$

$$= 132 + 180$$

= 312 ft, as before (when we assumed a perception time of ½ sec and reaction time of 1 sec).

In conclusion, this article has demonstrated how to compute the stopping distance of a car traveling at a given speed-- a real life application of quadratic equations, and science topics such as motion, kinetic energy, and work done. It has also discussed the idea that although mathematics can be utilized to model real life situations, there is much more to consider, if one wants more refined and appropriate solutions to real life problems. For readers interested in exploring safe following-distance (such as the 2-second rule), and the importance of reaction time, please refer to the **Appendix, Note 4**. For ideas on using the formulas given here on stopping distances as a source of exercises in solving quadratic equations, please refer to the **Appendix, Note 3**.

Appendix

Note 1 (about the approximations, $s = \frac{u^2}{20}$, and $s = u + u^2/20$)

The approximation $s = \frac{u^2}{20}$, where s is the braking distance in feet (that is, the distance traveled by the car before it comes to a stop, after application of the brakes), and u is the speed in mph where the brakes were first applied, was one that I found from a number of sources I researched. Interestingly enough, the equation does give an approximation (numerical) of the *braking distance* found at various sites related to stopping distance, including those from automobile manufacturers, who usually state, inaccurately, that that is the *stopping distance*, when it is actually the *braking distance*.

BTW, such careless terminology (using “stopping distance” when “braking distance” is meant) is not limited to car manufacturers, but is also evident in the British Highway Code, where the “thinking distance” is equated to “reaction time distance”—when, in reality the “thinking distance” is a combination of both “perception time distance” (distance traveled after perceiving the need to apply the brakes) and “reaction time distance” (distance traveled after reacting to the danger, for example, in the time taken to, say, move the leg from the accelerator pedal to the brake, for cars with manual transmission).

The British Highway Code seems to use the approximation
$s = u + u^2/20$, where u is in mph, and s is in ft which gives, for u = 60 mph,
$s = 60 + 60^2/20 = (60 + 180)$ ft = 240 ft, where the 60 ft is considered (erroneously) the “thinking distance,” when, in fact, it is the “reaction distance.”
If *stopping distance* = *perception distance* + *reaction distance* + *braking distance*, then using the approximation $s = u + u^2/20$, for the *stopping distance*, it only gives the *reaction distance* plus the *braking distance*, and is ignoring the perception distance, thereby resulting in a much lower numerical value for the stopping distance. For example, if it takes about 0.7 sec to perceive the danger when traveling at 60 mph (88 fps), then the perception distance would be 0.7 x 88 = 61.6 ft, and this 61.6 ft would need to be added to the 240 ft, to get a more realistic (approximate) stopping distance of 301.6 ft.

Note 2 (about the approximation $s = 2.2\,u + u^2/20$)

For the approximation $s = 2.2\,u + u^2/20$, where s = stopping distance in ft, and u = initial speed in mph,
I used the following equation, found on p.28 of the following report, based on sources from AASHTO (American Association of State Highway and Transportation Officials) that drew upon data collected experimentally:
Discussion Paper No. 8.A, STOPPING SIGHT DISTANCE AND DECISION SIGHT DISTANCE

prepared for Oregon Department of Transportation, Salem, Oregon
by the Transportation Research Institute, Oregon State University, Corvallis, Oregon 97331-4304
February 1997, accessed on March 3, 2012 at

http://www.oregon.gov/ODOT/HWY/ACCESSMGT/docs/StopDist.pdf?ga=t

SSD = 1.47 Vt +V^2/[30 (f±g)]   (ENGLISH units)
where SSD = required stopping sight distance, ft.
V = speed, mph
t = perception-reaction time, sec (that is, the "thinking time")
f = coefficient of friction
g = grade, decimal

I used t =1 sec (for the thinking time = perception time + reaction time) in the equation above, and converted the V mph to fps by multiplying by 5280/3600 (but did not convert V^2), and substituted u for V, and s for SSD, used f =0.67 for a dry surface, and omitted the grade/gradient g, as shown below:
SSD = 1.47 Vt +V^2/[30 (f±g)] becomes
s=1.47 ut +u^2/[30 f], omitting g
s = 1.47 (u x 5280/3600) (1) + u^2/[30 x 0.67], converting the u mph to fps, and using t =1 sec, & f = 0.67
s = 2.2 u + u^2/20
So, for u = 60 mph,
s = 2.2 (60) + 60^2/20
 = 132 + 180 = 312 ft, as the stopping distance.

Note 3 (about safe following distance and reaction time implications)
For those interested in the following distance, or the distance behind a car you are following on the road to minimize the danger of colliding, the following URL gives some useful information:

http://www.driveandstayalive.com/info%20section/following-distances.htm

Although the 2 second rule is given and explained, a further caveat is in order: if one were to estimate the 2 seconds by counting one-thousand-one, one-thousand-two, by starting to count as soon as the vehicle that is ahead of you passes a fixed object, and you reach the same spot that the vehicle was at 2 seconds ago (as suggested), one must realize that it is difficult to focus on that and, at the same time look out for any sudden hazard or sudden braking by the vehicle ahead. So, to be on the safe side, it might be better to keep a look out for brake lights of the car ahead, go slower, stay back a little, and remain alert for a sudden stop.

BTW, that site also has a table of braking and stopping distances, that seems to be a result of using the British Highway Code approximation $s = u + u^2/20$.

Another useful reference on safe driving, Larry Woolfe's "Staying Alive: The Physics, Mathematics, and Engineering of Safe Driving" at http://www.sci-ed-ga.org/modules/driving/ has many important concepts of safe driving, and the chapter on reaction time and speed at which one is travelling, accessible at http://www.sci-ed-ga.org/modules/driving/parts/investigation6_.pdf has some instructive examples of why a driver needs to be aware of reaction time and the speed at which one is traveling, and how these affect safety on the road. For example, the average reaction time is stated to be 1.5 sec, meaning a distance of 1.5 x 88 ft = 132 ft would have been covered if the speed were 60 mph, but that for a person who has taken some drinks, the reaction time is doubled to almost 3 sec, meaning the distance traveled would now be doubled, too, to 264 ft, under the same speed. In other words, *the reaction time is going to be much more crucial to safety, than merely the stopping distance, if one is following a car*.

Note 4 (about the accuracy of published stopping distances, and factors affecting stopping distance).

To those interested in finding out whether government published stopping distances are realistic, there is an instructive site showing stopping distances computed in Australia, for different cars, at
http://www.sdt.com.au/safedrive-directory-STOPPINGDISTANCE.htm
This site claims that it has been found that most cars nowadays can come to a stop within 40 m (about 131 ft), about half the distance shown in "official" publications (assuming an initial speed of 100 kph or about 60 mph). This could be because of improved braking capabilities such as more efficient hydraulic braking systems, ABS, etc. But it is also true that it is better to err on the side of safety, as factors other than a car's improved braking capabilities come into play when required to make a sudden stop. Some of these factors are also found at this site, and among them are the following: human perception time, human reaction time, vehicle reaction time, and vehicle braking capability (such as type of braking system, tire pressure, coefficient of friction, etc).

Note 5 (about using the approximation $s = u + u^2/20$ as a resource for practicing the solving of quadratic equations).
For those interested in using the approximation $s = u + u^2/20$ for practice in solving quadratic equations, it might be beneficial to *give a few stopping distances, s, and ask students to compute the speed at which the car might have been traveling*. For example, if the stopping distance is 240 ft, then, students need to solve the following quadratic equation:

$240 = u + u^2/20$, which can be re-written as $u^2 + 20\,u - 20\,(240) = 0$, resulting in $u^2 + 20\,u - 4800 = 0$, which can then be solved by factorizing thus: $(u - 60)\,(u + 80) = 0$, resulting in $u = 60$ mph.

Then students could be asked to compute the speeds corresponding to different stopping distances such as the following: 75 ft, 120 ft, 175 ft, and 315 ft, all of which result in quadratic equations that can be factored, with solutions, 30 mph, 40 mph, 50 mph, and 70 mph, respectively. Of course, then students could then be given distances that do not result in quadratic equations that can be factored, in which case they can then use the quadratic formula (or completing the square) to solve those equations.

As further practice, the approximation $s = 2.2\,u + u^2/20$ could also be used, given different values of s.

But, please note that, in real life cases, it is not easy to get accurate measurements of the stopping distance, and measurement of **skid marks at an accident do not give the actual stopping distance**. The skid marks show a distance much **less** than the actual stopping distance, and other factors such as drag factor (tire and road surface interface), braking efficiency, etc, need to be taken into account as shown at http://www.harristechnical.com/articles/skidmarks.pdf , and there are other quadratic equations involved, rather than the ones we have discussed in this paper.